Null Energy Condition Violations in Eternal Inflation*

Serge Winitzki

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Department of Physics and Astronomy, Tufts University, Medford, MA 02155

Abstract

The usual scenario of "eternal inflation" involves an approximately de Sitter spacetime undergoing upward fluctuations of the local expansion rate H. This spacetime requires frequent violations of the Null Energy Condition (NEC). We investigate the fluctuations of the energy-momentum tensor of the scalar field in de Sitter space as a possible source of such violations. We find that fluctuations of the energy-momentum tensor smeared in space and time are well-defined and may provide the NEC violations. Our results for slow-roll inflation are consistent with the standard calculations of inflationary density fluctuations. In the diffusive regime where quantum fluctuations dominate the slow-roll evolution, the magnitude of smeared energy-momentum tensor fluctuations is large enough to create frequent NEC violations.

The usual picture of eternal inflation [2] involves a locally FRW spacetime which is well described by a de Sitter-like metric with a slowly varying expansion rate H,

$$ds^2 = dt^2 - [a(x,t)]^2 \, d\vec{x}^2, \; a(x,t) \approx e^{\int H(x,t)dt}. \tag{1}$$

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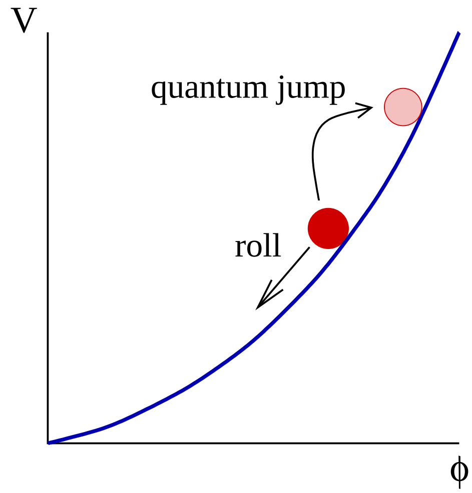

Figure 1: A scalar field rolls down the potential slowly and causes inflation. The eternal inflation argument is that occasional quantum jumps in localized regions can raise the scalar field to a higher value of the energy density, leading to regions of faster inflation.

In a universe dominated by the vacuum energy $V(\phi)$ of the scalar field ϕ (the inflaton), the equations of motion in the slow roll approximation are

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv [H(\phi)]^2 = \frac{8\pi}{3} V(\phi), \tag{2}$$

$$\dot{\phi} = -\frac{1}{4\pi}\frac{dH(\phi)}{d\phi} \tag{3}$$

($G = c = 1$ in our units). According to the standard picture, quantum fluctuations of the inflaton on super-horizon scales $L \gtrsim H^{-1}$ behave classically and lead to a random variation of the field $\phi(x,t)$ and of the expansion rate $H(x,t)$ on these scales L. This modifies Eq. (3) so that, in addition to the deterministic change, the field randomly jumps by $\delta\phi \sim H/(2\pi)$ in horizon-size regions during one Hubble time H^{-1} (see Fig. 1). Occasionally, the field will move upward on the potential, delaying the end of inflation. These random "upward jumps" create regions with a larger expansion rate.

For most inflaton potentials, there generically exists a regime where the "jumps" significantly affect or even dominate the scalar field evolution. The

latter happens for values of ϕ such that

$$\left|\dot{\phi}\right| \ll \frac{\left(H\left(\phi\right)\right)^2}{2\pi}. \tag{4}$$

If inflation starts with these values of ϕ, the universe will contain a statistical distribution of inflating regions, some of which will thermalize; in that case inflation becomes "eternal" because at any (arbitrarily late) time there will be some regions which have not yet reached thermalization. In this picture of eternal inflation, upward jumps of the effective expansion rate H in horizon-size regions are assumed to be caused by backreaction of fluctuations of the scalar field ϕ on the metric.

It has been known that eternal inflation requires violations of the null energy condition [3]. The Einstein equation

$$\dot{H} = -4\pi G\left(\rho + p\right) + \frac{k}{a^2} \tag{5}$$

shows that if H is to increase and the curvature is zero or negative ($k \leq 0$), the term $\rho + p$ must be negative. For the standard energy-momentum tensor $T_{\mu\nu}$ of a homogeneous fluid, this translates into the null energy condition (NEC),

$$T_{\mu\nu}N^\mu N^\nu < 0 \tag{6}$$

for some null vector N^μ. In other words, the NEC must be violated in any patch of spacetime of size $L \gtrsim H^{-1}$ that undergoes an upward jump in its local expansion rate H.

The issue we are concerned with is the source of these NEC violations during eternal inflation. Energy density of a classical scalar field cannot violate the NEC, since

$$T_{\mu\nu}N^\mu N^\nu = \left(N^\mu \partial_\mu \phi\right)^2 \geq 0. \tag{7}$$

A quantum field $\hat{\phi}$ may violate the NEC in some quantum states; however, the expectation value of the (renormalized) energy-momentum tensor $\hat{T}^{ren}_{\mu\nu}$ of a scalar field in a de Sitter-invariant state does not cause violations of the NEC. In such a state

$$\left\langle \hat{T}^{ren}_{\mu\nu} \right\rangle \propto g_{\mu\nu} \tag{8}$$

because of de Sitter invariance, and therefore the NEC operator has vanishing expectation value,

$$\left\langle \hat{O}^{ren} \right\rangle \equiv \left\langle \hat{T}^{ren}_{\mu\nu} N^\mu N^\nu \right\rangle = 0. \tag{9}$$

Even if the expectation value of $\hat{T}_{\mu\nu}$ does not violate the NEC, the energy-momentum tensor has fluctuations around the mean which may be significant. In the picture of semiclassical gravity, the spacetime metric is completely determined by the expectation value of the energy-momentum tensor and is insensitive to its fluctuations. We would like to include the effect of energy-momentum fluctuations on the metric and for that we would need to go beyond semiclassical gravity. In the present work we restrict ourselves to an evaluation of $\langle \hat{O}^2 \rangle$ and do not consider the effect of the fluctuations on the metric, which is a separate and more complicated issue.

A direct evaluation of the fluctuations of $\hat{T}_{\mu\nu}$ is problematic since the two-point function of the energy-momentum tensor $\left\langle \hat{T}_{\mu\nu}(x)\, \hat{T}_{\rho\sigma}(x) \right\rangle$ diverges at coincident points x. Even if this divergence were somehow to be renormalized away, any remaining finite piece will have to be of the form

$$\propto g_{\mu\nu} g_{\rho\sigma} + (\text{permutations}) \tag{10}$$

and, after a contraction with $N^\mu N^\nu N^\rho N^\sigma$, will not contribute to the NEC.

From the formal point of view, the renormalized energy-momentum tensor $\hat{T}^{ren}_{\mu\nu}$ and the NEC $\hat{O}^{ren}$ are not operators but operator-valued distributions that become well-defined operators after averaging with a window function, for instance

$$\hat{O}^{ren}_W \equiv \int d^4x \sqrt{-g}\, W\left(\frac{x}{L}\right) \hat{O}^{ren}. \tag{11}$$

Here the window function profile $W(x)$ is chosen so that it falls off rapidly and provides a smearing on scale L. Averaging over time as well as over space is necessary to avoid divergences in $\langle (\hat{O}^{ren}_W)^2 \rangle$. (Details of the calculations will be given in [1]). We shall adopt the viewpoint that the quantum energy-momentum tensor smeared in space and time on scales $L \gtrsim H^{-1}$ behaves quasi-classically and may provide NEC violations in certain horizon-size regions. This is similar to the assumption made in usual calculations of inflationary perturbations where a coarse-graining on super-horizon scales is performed.

We take the quantum state of the field $\hat{\phi}$ during slow-roll inflation to be a superposition

$$\hat{\phi}(x,t) = \phi_0(t) + \delta\hat{\phi}(x,t), \tag{12}$$

where $\phi_0(t)$ is the classical slow-roll trajectory (represented by a suitable coherent state) and $\delta\hat{\phi}$ is a quantum scalar field in the usual Bunch-Davies vacuum state of the approximately de Sitter spacetime. We have computed

the expectation value $\left\langle \hat{O} \right\rangle$ and the dispersion $\left\langle \hat{O}^2 \right\rangle$ of the operator given by Eq. (11) in this quantum state. Although the expectation value $\left\langle \hat{O} \right\rangle > 0$ in this state, the NEC may be frequently violated if $\left\langle \hat{O}^2 \right\rangle \gg \left\langle \hat{O} \right\rangle^2$.

The calculations were made for a massless field $\delta\hat{\phi}$ in de Sitter space, with a constant null vector field

$$N^\mu \propto (H\eta)^2 \left[1, \mathbf{n}\right], \ \mathbf{n} = const, \tag{13}$$

and an arbitrary normalized window profile $W(x)$ that decays quickly (it is sufficient to assume exponential decay) for $|x| \gtrsim 1$. The smearing scales of interest are $L = (\varepsilon H)^{-1}$ for both space and time (with $\varepsilon \lesssim 1$). Under these assumptions we have obtained the ratio of dispersion to mean,

$$\frac{\left\langle \hat{O}^2 \right\rangle}{\left\langle \hat{O} \right\rangle^2} = \left(\frac{H^2}{2\pi\dot{\phi}_0}\right)^2 \max\left(c_1\varepsilon^2, c_2\varepsilon^4\right) + \left(\frac{H^2}{2\pi\dot{\phi}_0}\right)^4 c_3\varepsilon^8 \tag{14}$$

where $c_1,\ c_2,\ c_3 \sim O(1)$ are window-dependent constants. (This result is otherwise insensitive to the choice of the window profile.) The first term on the RHS of Eq. (14) corresponds to first-order terms in the perturbative expansion in $\delta\phi$, while the second term comes from second-order expansion terms.

Within our assumptions, it follows from Eq. (14) that fluctuations of the energy-momentum tensor lead to frequent NEC violations when

$$\left|\dot{\phi}_0\right| \ll \frac{H^2}{2\pi}. \tag{15}$$

Note that this is the same condition as Eq. (4) for fluctuation domination. If this condition holds, the second term on the RHS of Eq. (14) is of the same order or greater than the first term; in other words, perturbation theory is inadequate in the fluctuation-dominated regime. In the opposite regime, perturbation theory is valid, the NEC holds and we obtain an agreement with standard calculations of inflationary perturbations.

We have shown that there exists a range of scalar field values ϕ where fluctuations of the smeared energy-momentum tensor on super-horizon scales are significant and suggest frequent NEC violations. This range of ϕ coincides with the regime where the quantum "jumps" dominate the evolution of ϕ, which is required for eternal inflation. In the traditional description of

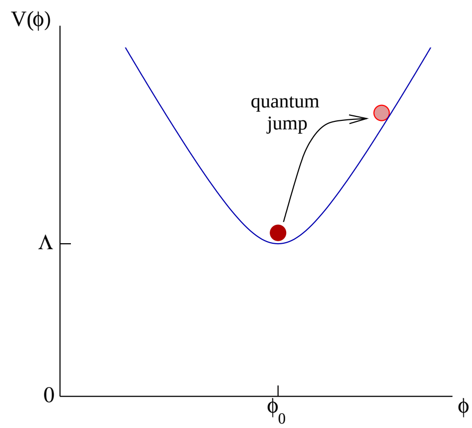

Figure 2: A classical scalar field with ground state at the bottom of the potential with non-zero energy density Λ causing de Sitter expansion. Smeared quantum fluctuations of the energy-momentum tensor may violate the NEC. In the heuristic picture, these violations would correspond to localized quantum jumps of the scalar field to higher values of the potential, causing faster inflation in those regions and an instability of de Sitter expansion towards eternal inflation.

inflation, the quantum field ϕ smeared on super-horizon scales is treated as a classical field. If super-horizon scale smearing is an accurate phenomenological description of the quantum-to-classical transition of the vacuum fluctuations in de Sitter space, these fluctuations would indeed seem to provide the necessary NEC violations.

Following the same approach, the present analysis suggests a possibility of spontaneous eternal inflation in de Sitter spacetime. Suppose we had a massive scalar field with the potential shown in Fig. 2. The classical field is located at the bottom of the potential and gives rise to a background inflation. The quantum state of the fluctuations of ϕ is assumed to be the Bunch-Davies vacuum. In this state there will also be fluctuations of the energy density and, since the expectation value $\left\langle \hat{O} \right\rangle = 0$ while $\left\langle \hat{O}^2 \right\rangle > 0$, violations of the NEC will happen. So the standard picture of eternal inflation picture would suggest that de Sitter space is destabilized by the presence of a scalar field in the Bunch-Davies vacuum.

One may also apply similar considerations to Minkowski spacetime and conclude, by setting $\Lambda = 0$ in Fig. 2, that Minkowski spacetime is unstable to eternal inflation. This conclusion would be contrary to the usual

assumption that flat spacetime is stable. However, there is a crucial difference between Minkowski spacetime and a de Sitter spacetime, namely that the "horizon scale" H^{-1} in Minkowski spacetime is infinite. Therefore for Minkowski spacetime to be unstable to eternal inflation, the NEC-violating fluctuation would have to survive for an infinitely long time. The probability for this to happen vanishes and hence Minkowski spacetime is stable towards eternal inflation. However, in FRW spacetimes with a finite horizon size we might still find an instability towards eternal inflation. To resolve this question, a more precise picture of the backreaction is needed.

An unsatisfactory feature of our argument is that it only concerns itself with the spacetime before and after the fluctuation. The evolution of the spacetime during the fluctuation itself is not considered. Our attempt to go beyond the semiclassical analysis by calculating fluctuations of the energy-momentum tensor was not entirely successful since we do not yet have a scheme for calculating the backreaction of NEC violating fluctuations on the metric. This issue requires further study.

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